Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00109-X

J. Appl. Maths Mechs, Vol. 64, No. 5, pp. 797–810, 2000

0021-8928/00/$—see front matter

INVESTIGATION OF THE STABILITY OF PERIODIC MOTIONS OF AN AUTONOMOUS HAMILTONIAN SYSTEM IN A CRITICAL CASE†

A. P. MARKEYEV

Moscow

(Received 18 February 2000)

The problem of the orbital stability of periodic motions of a Hamiltonian system with two degrees of freedom is considered. The Hamilton function does not depend explicitly on the time and is analytic in the neighbourhood of the trajectory of the unperturbed motion. The critical case, when all the multipliers are real and have moduli equal to unity, is investigated. The stability and instability conditions are obtained using Lyapunov's second method and the KAM theory. Constructive algorithms for checking these conditions are given. The case of a system containing a small parameter is considered in particular. On the technical side, the investigation rests primarily on the classical theory of perturbations of Hamiltonian systems and its modern modifications. The problem of the stability of the permanent rotation of a heavy circular disc which is in collision with a fixed horizontal plane and the problem of the stability of the plane rotations of a rigid body about a fixed point are considered as applications.

1. THE HAMILTON FUNCTION AND ITS NORMAL FORM

Suppose an autonomous Hamiltonian system with two degrees of freedoms has a periodic motion and in a neighbourhood of a closed trajectory of phase space corresponding to this motion the Hamilton function is analytic. We will assume, without loss of generality, that the period is equal to 2π.

The canonically conjugate variables ξ_i, η_i (ξ_i are the coordinates and η_i are the momenta, $i = 1, 2$) can be chosen [1] such that the solution corresponding to the periodic motion considered can be written in the form

$$\xi_1(t) = t + \xi_1(0), \quad \eta_1 = \xi_2 = \eta_2 = 0 \tag{1.1}$$

Here the Hamiltonian Γ will be 2π-periodic in ξ. It can be expanded in a converging series in powers of η_1, ξ_2, η_2:

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + \ldots + \Gamma_k + \ldots \tag{1.2}$$

where Γ_k is the form of the power of k relative to $|\eta_1|^{1/2}$, ξ_2, η_2. We will write the forms of the second, third and fourth power, required later, in the form

$$\begin{aligned} &\Gamma_2 = \eta_1 + \varphi_2(\xi_2, \eta_2, \xi_1), \quad \Gamma_3 = \psi_1(\xi_2, \eta_2, \xi_1)\eta_1 + \varphi_3(\xi_2, \eta_2, \xi_1) \\ &\Gamma_4 = \chi(\xi_1)\eta_1^2 + \psi_2(\xi_2, \eta_2, \xi_1)\eta_1 + \varphi_4(\xi_2, \eta_2, \xi_1) \end{aligned} \tag{1.3}$$

where $\chi(\xi_1)$ is a 2π-periodic function of ξ_1 and φ_m, ψ_m are forms of power m in ξ_2, η_2 with coefficients that are 2π-periodic in ξ_i.

The orbital stability of the unperturbed motion (1.1) denotes stability of the system with Hamiltonian (1.2) with respect to perturbations of the quantities η_1, ξ_2, η_2.

Two of the multipliers of the system of equations of the unperturbed motion, linearized in the neighbourhood of periodic motion (1.1), are equal to unity, and the other two are the roots of the equation

$$\rho^2 - 2A\rho + 1 = 0 \tag{1.4}$$

where $2A = x_{11}(2\pi) + x_{22}(2\pi)$ and $x_{ii}(2\pi)$ are the elements of the matrix $\mathbf{X}(\xi_1)$, calculated for $\xi_1 = 2\pi$, of the fundamental solutions $(\mathbf{X}(0)) = \mathbf{E}$, where $\mathbf{E}$ is the second-order identity matrix) of the linear system with coefficients that are 2π-periodic with respect to the independent variable ξ_1

†Prikl. Mat. Mekh. Vol. 64, No. 5, pp. 833–847, 2000.

$$\frac{d\xi_2}{d\xi_1} = \frac{\partial\varphi_2}{\partial\eta_2}, \quad \frac{d\eta_2}{d\xi_1} = -\frac{\partial\varphi_2}{\partial\xi_2} \tag{1.5}$$

Here φ_2 is the part of the function Γ_2 from (1.3) that is quadratic in ξ_2, η_2.

We will investigate the critical case when $|A| = 1$, i.e. when Eq. (1.4) has real roots (when $A = -1$) or roots $\rho_1 = \rho_2 = 1$ (when $A = 1$). Confining ourselves to the case of the general situation, we will assume that the matrix $\mathbf{X}(2\pi)$ does not reduce to diagonal form. In this case, the unperturbed periodic motion is orbitally unstable in the linear approximation.

To investigate the non-linear problem of the orbital stability of periodic motion (1.1) it is best to obtain the normal form of the Hamilton function (1.2) using a canonical replacement of variables. To do this we will first use a real linear univalent canonical replacement of variables, periodic in ξ_1

$$\xi_2 = n_{11}(\xi_1)u_2 + n_{12}(\xi_1)\upsilon_2, \quad \eta_2 = n_{21}(\xi_1)u_2 + n_{22}(\xi_1)\upsilon_2 \tag{1.6}$$

to reduce the quadratic Hamiltonian φ_2 of Eqs (1.5) to its normal form $1/2\ \delta\upsilon_2^2$, where $\delta = 1$ or -1, its specific value being determined when constructing replacement (1.6).

If $A = 1$, we can obtain a matric N of the replacement of variables (1.6) that is 2π-periodic in ξ_1. It has the form [2]

$$\mathbf{N} = \mathbf{X}(\xi_1)\mathbf{P}\mathbf{Q}(\xi_1) \tag{1.7}$$

where

$$\mathbf{Q} = \begin{Vmatrix} 1 & -\delta\xi_1 \\ 0 & 1 \end{Vmatrix} \tag{1.8}$$

and the number δ and the matrix $\mathbf{P}$ are defined by the following formulae: if $x_{12}(2\pi) \neq 0$, we have

$$\mathbf{P} = \begin{Vmatrix} a & 0 \\ b & a^{-1} \end{Vmatrix}, \quad \delta = \operatorname{sign} x_{12}(2\pi) \tag{1.9}$$

$$a = |x_{12}(2\pi)/(2\pi)|^{1/2}, \quad b = \delta(x_{22}(2\pi) - 1)|2\pi x_{12}(2\pi)|^{-1/2}$$

and if $x_{21}(2\pi) \neq 0$, we have

$$\mathbf{P} = \begin{Vmatrix} d & c^{-1} \\ -c & 0 \end{Vmatrix}, \quad \delta = -\operatorname{sign} x_{21}(2\pi)$$

$$c = |x_{21}(2\pi)/(2\pi)|^{1/2}, \quad d = \delta(x_{11}(2\pi) - 1)|2\pi x_{21}(2\pi)|^{-1/2} \tag{1.10}$$

When $A = -1$ replacement (1.6) will be 4π-periodic in ξ_1. It can be obtained in the form of the product of three matrices (1.7), where $\mathbf{Q}$ is given by formula (1.8) and $\mathbf{P}$ is defined by Eqs (1.9) or (1.10), in which a and c remain as before, while

$$b = -\delta(x_{22}(2\pi) + 1)|2\pi x_{12}(2\pi)|^{-1/2}, \quad d = -\delta(x_{11}(2\pi) + 1)|2\pi x_{21}(2\pi)|^{-1/2}$$

where $\delta = -\operatorname{sign} x_{12}(2\pi)$ if $x_{12}(2\pi) \neq 0$ and $\delta = \operatorname{sign} x_{21}(2\pi)$ if $x_{21}(2\pi) \neq 0$.

If $S(\upsilon_2, \xi_2, \xi_1)$ is the generating function of transformation (1.6), then, by the theory of canonical transformations [3], we have the identity

$$\frac{\partial S}{\partial\xi_1} + \varphi_2 = \frac{1}{2}\delta\upsilon_2^2 \tag{1.11}$$

On the left-hand side of this identity the quantities ξ_2, η_2 are expressed in terms of u_2 and υ_2 by formulae (1.6).

To obtain the transformation $\xi_1, \xi_2, \eta_1, \eta_2 \to u_1, u_2, \upsilon_1, \upsilon_2$, canonical with respect to all the variables and which reduces the function Γ_2 from (1.2) to normal form, we take a generating function of the form $\upsilon_1\xi_1 + S$. Then $u_1 = \xi_1$, and

$$\eta_1 = v_1 + \frac{\partial S}{\partial \xi_1} = v_1 + k_{20}u_2^2 + k_{11}u_2v_2 + k_{02}v_2^2 \tag{1.12}$$

where k_{ij} are periodic functions of ξ_1, found from identity (1.11).

The Hamilton function (1.2) can be written as follows in the new variables u_i, υ_i

$$F = F_2 + F_3 + F_4 \ldots + F_k + \ldots \tag{1.13}$$

$$F_2 = v_1 + \tfrac{1}{2}\delta v_2^2,\quad F_3 = f_1(u_2, v_2, u_1)v_1 + f_3(u_2, v_2, u_1) \tag{1.14}$$

$$F_4 = \chi(u_1)v_1^2 + f_2(u_2, v_2, u_1)v_1 + f_4(u_2, v_2, u_1)$$

Here f_k is a form of the power k in u_2, υ_2

$$f_k = \sum_{\nu+\mu=k} f_{\nu\mu}(u_1)u_2^\nu v_2^\mu$$

The coefficients $f_{\nu\mu}$ have period τ in u_1. The value of τ is equal to 2π when $A = 1$ in Eq. (1.4) and $\tau = 4\pi$ when $A = -1$. Expressions for $f_{\nu\mu}$ in terms of the coefficients of Hamiltonian (1.2) follow from the identities $\psi_1\eta_1 + \varphi_3 = F_3$, $\chi\eta_1^2 + \psi_2\eta_1 + \varphi_4 = F_4$, on the left-hand sides of which ξ_2, η_2 and η_1 are replaced, in accordance with formulae (1.6) and (1.12), while $\xi_1 = u_1$.

To normalize terms of the third and higher powers in Hamilton function (1.3) we will use the Depri–Hori method [4,5]. We will confine ourselves to considering terms up to the fourth power inclusive. In a sufficiently small neighbourhood of the unperturbed periodic motion, the normalizing canonical transformation u_1, υ_1, u_2, υ_2, $\to w_1$, r_1, q_2, p_2, can be obtained close to the identity transformation

$$u_1 = w_1 + \ldots,\quad v_1 = r_1 + \ldots, u_2 = q_2 + \ldots, v_2 = p_2 + \ldots$$

where the dots converging series in powers of r_1, q_2 and p_2 with coefficients τ-periodic in w_1. The normalized Hamiltonian (1.13) takes the form

$$H = r_1 + \tfrac{1}{2}\delta p_2^2 + h_{30}q_2^3 + h_{10}q_2r_1 + h_{40}q_2^4 + h_{20}q_2^2r_1 + h_{00}r_1^2 + O_5 \tag{1.15}$$

where h_{ij} are constant quantities while O_5 is a series which begins with terms no less than the fifth power in $|r_1|^{1/2}$, q_2, p_2, and the coefficients of the series τ are τ-periodic in the variable w_1. Orbital stability of the unperturbed periodic motion (1.1) is equivalent to stability of a system with normalized Hamiltonian (1.15) with respect to perturbations of the quantities r_1, q_2 and p_2.

Omitting the fairly lengthy calculations, we will write the final formulae required to calculate coefficients h_{ij} of the normal form. We have

$$h_{30} = \langle f_{30}\rangle,\ h_{10} = \langle f_{10}\rangle \tag{1.16}$$

$$h_{40} = \left\langle f_{40} - \tilde{f}_{30}\tilde{f}_{10} + \tfrac{3}{2}(\tilde{f}_{30}\tilde{w}_{21} - \tilde{f}_{21}\tilde{w}_{30})\right\rangle + \tfrac{1}{2}\delta(3\langle f_{30}\rangle\langle f_{12}\rangle - \langle f_{21}\rangle^2) \tag{1.17}$$

$$h_{20} = \left\langle f_{20} - \tilde{f}_{10}^2 + \tfrac{3}{2}(\tilde{f}_{30}\tilde{w}_{01} - \tilde{f}_{01}\tilde{w}_{30}) + \tfrac{1}{2}(\tilde{f}_{10}\tilde{w}_{21} - \tilde{f}_{21}\tilde{w}_{10}\right\rangle +$$

$$+\tfrac{1}{2}\delta(\langle f_{10}\rangle\langle f_{12}\rangle - 2\langle f_{21}\rangle\langle f_{01}\rangle) \tag{1.18}$$

$$h_{00} = \left\langle \chi + \tfrac{1}{2}(\tilde{f}_{10}\tilde{w}_{01} - \tilde{f}_{01}\tilde{w}_{10})\right\rangle - \tfrac{1}{2}\delta\langle f_{01}\rangle^2 \tag{1.19}$$

Here we have used the notation u_1 for the coefficients $f_{\nu\mu}$ of forms (1.14), τ-periodic in u_1, where $f_{\nu\mu} = \langle f_{\nu\mu}\rangle + \tilde{f}_{\nu\mu}$, is the mean value of the function $\langle f_{\nu\mu}\rangle$ over the period. Similar notation is also used for the coefficients $w_{\nu\mu}$ of the expansion in series of the generating function of the Lie transformation in the Depri–Hori method. The functions $\tilde{w}_{\nu,\mu}$ occurring in (1.17)–(1.19) satisfy the following system of differential equations

$$\frac{d\tilde{w}_{30}}{du_1} = \tilde{f}_{30},\quad \frac{d\tilde{w}_{21}}{du_1} = \tilde{f}_{21} - 3\delta\tilde{w}_{30}$$

$$\frac{d\tilde{w}_{10}}{du_1} = \tilde{f}_{10}, \quad \frac{d\tilde{w}_{01}}{du_1} = \tilde{f}_{01} - \delta\tilde{w}_{10}$$

The initial values of the functions $\tilde{w}_{\nu,\mu}$ can obviously be chosen so that the mean values $\langle\tilde{w}_{\nu,\mu}\rangle$ are equal to zero.

2. THE INSTABILITY OF THE PERIODIC MOTION

In this section we will prove the following assertion.

Theorem 1. If the coefficient h_{30} of normal form (1.15) is non-zero or $h_{30} = 0$ but $\delta h_{40} < 0$, the periodic motion is orbitally unstable.

In order to prove the correctness of the theorem, it is sufficient to demonstrate the instability for values of the perturbations r_1, q_2 and p_2, belonging to the energy level $H = 0$, on which the perturbed motion (1.1) lies. On this energy level the motion of the system is described by Whittaker's equations, which have the form of the Hamilton equations [3]. The function $K(q_2, p_2, w_1)$, where $r_1 = -K$ is the root of the equation $H = 0$, plays the role of the Hamilton function, while the quantity w_1 plays the role of the independent variable. In a sufficiently small neighbourhood of the unperturbed motion (1.1) the quantity w_1 increases monotonically and can play the role of time in the stability problem.

Suppose h_{30} is non-zero in (1.15). Solving the equation $H = 0$ for r_1, we obtain

$$K = \tfrac{1}{2}\delta p_2^2 + h_{30}q_2^3 - \tfrac{1}{2}\delta h_{10}q_2p_2^2 + O_4 \tag{2.1}$$

where O_4 is the set of terms higher than the third power in q_2 and p_2.

We will use Lyapunov's theorem on instability [6]. We will first simplify Hamiltonian (2.1) by making two canonical replacements of variables in succession. The first replacement $q_2, p_2 \to q_2', p_2'$ is univalent and is specified by the generating function

$$S_1 = q_2p_2' + \tfrac{1}{4}h_{10}q_2^2p_2'$$

while the second has valency δh_{30}^2 and has the form

$$q_2' = \delta h_{30}^{-1}Q_2, \quad p_2' = h_{30}^{-1}P_2$$

After these replacements the motion on the energy level $H = 0$ is described by the transformed Hamiltonian (2.1), which has the form

$$K = \tfrac{1}{2}P_2^2 + Q_2^3 + O_4 \tag{2.2}$$

We take the following function as the Lyapunov function

$$V = (Q_2 - \tfrac{1}{3})P_2$$

By virtue of the equations of motion with Hamiltonian (2.2) its derivative will be

$$\frac{dV}{dw_1} = Q_2^2 + P_2^2 + O_3$$

Since the derivative is positive-definite, while the function V itself is sign-variable, then, by Lyapunov's theorem, there is instability

Suppose now $h_{30} = 0$ and $\delta h_{40} < 0$. We will show that there is instability, as above, by using Lyapunov's theorem. The motion on the energy level $H = 0$ is described by equations to which the following Hamiltonian corresponds

$$K = \tfrac{1}{2}\delta p_2^2 - \tfrac{1}{2}\delta h_{10}q_2p_2^2 + h_{40}q_2^4 - \tfrac{1}{2}\delta h_{20}q_2^2p_2^2 + \tfrac{1}{4}h_{00}p_2^4 + O_5 \tag{2.3}$$

After the canonical univalent replacement of variables $q_2, p_2 \to q_2', p_2'$, defined by the generating function

$$S_2 = q_2p_2' + \tfrac{1}{4}h_{10}q_2^2p_2' + 1/24(3h_{10}^2 + 4h_{20})q_2^3p_2' - \tfrac{1}{4}\delta h_{00}q_2p_2'^3$$

and a subsequent canonical replacement (with valency $\delta|h_{40}|$)

$$q_2' = \delta|h_{40}|^{-1/2}Q_2, \quad p_2' = |h_{40}|^{-1/2}P_2$$

Hamiltonian (2.3) takes the form

$$K = \tfrac{1}{2}P_2^2 - Q_2^4 + O_5 \tag{2.4}$$

Suppose $V = Q_2P_2$. Then, by virtue of equations corresponding to (2.4), we obtain

$$\frac{dV}{dw_1} = 4Q_2^4 + P_2^2 + O_5$$

Since V is sign-variable, while dV/dw_1 is positive-definite function, then, by Lyapunov' theorem, there is instability.

Theorem 1 is proved.

3. THE CONDITION FOR ORBITAL STABILITY OF PERIODIC MOTION

The following theorem, which gives the sufficient condition for the orbital stability of unperturbed motion, holds.

Theorem 2. If the coefficient h_{30} in normal form (1.15) is equal to zero, but then $\delta h_{40} > 0$, periodic motion (1.1) is orbitally stable.

We will prove this theorem by the methods of the KAM theory [7]. We will carry out the proof in several stages.

3.1. *The introduction of a small parameter.* Using the smallness of the quantities r_1, q_2 and p_2, we will introduce a small parameter $\varepsilon(0 < \varepsilon \ll 1)$ and we will slightly transform Hamiltonian (1.15) using the following canonical replacement of variables (with valency $\varepsilon^{-3/2}\delta|h_{40}|$)

$$u_1 = U_1, \quad r_1 = \varepsilon^{3/2}\delta|h_{40}|^{-1}R_1 \tag{3.1}$$

$$q_2 = \varepsilon^{1/2}\delta|h_{40}|^{-1/2}Q_2, \quad p_2 = \varepsilon|h_{40}|^{-1/2}P_2$$

In the new variables the perturbed motion has a Hamilton function of the form

$$H = R_1 + \varepsilon^{1/2}\Phi(Q_2, P_2, R_1) + \varepsilon\Phi_1(Q_2, P_2, R_1, u_1, \varepsilon^{1/2}) \tag{3.2}$$

where

$$\Phi = \tfrac{1}{2}P_2^2 + Q_2^4 + \alpha Q_2R_1 \quad (\alpha = \delta h_{10}|h_{40}|^{-1/2}) \tag{3.3}$$

while the function Φ_1 is analytic with respect to all its arguments and i τ-periodic in u_1.

3.2. *The approximate system.* If we drop the term $\varepsilon\Phi_1$ in (3.2), we obtain an approximate system having the Hamiltonian

$$H^* = R_1 + \varepsilon^{1/2}\Phi(Q_2, P_2, R_1) \tag{3.4}$$

In the approximate system the quantity R_1 is constant and the change of the variables Q_2 and P_2 with time is described by canonical equations with Hamilton function

$$\Psi = \varepsilon^{1/2}\Phi(Q_2, P_2, R_1) \tag{3.5}$$

where R_1 is a constant parameter. These equations have the integral

$$\Phi(Q_2, P_2, R_1) = h = \text{const} \tag{3.6}$$

For actual motion $h \geqslant h^0$ $-(27/256)^{1/3}(\alpha R_1)^{4/3}$. When $h = h^0$ we have stable equilibrium for which $P_2 = 0$ and $Q_2 = \mp(1/4|\alpha R_1|)^{1/3}$, where the upper sign corresponds to the case $\alpha R_1 \geqslant 0$ and the lower sign corresponds to the case $\alpha R_1 \leqslant 0$. When $h > h^0$ oscillations occur in the neighbourhood of this equilibrium position.

Phase portraits of the system with Hamiltonian (3.5) are shown in Fig. 1, Figures 1(a), (b) and (c) correspond to the cases $\alpha R_1 > 0$, $\alpha R_1 = 0$ and $\alpha R_1 < 0$, respectively.

The point $Q_2 = P_2 = 0$ lies inside the region enveloped by phase curve (3.6) if $h > 0$ on this curve. The origin of coordinates $Q_2 = P_2 = 0$ itself corresponds to the case $h = 0$ in Fig. 1(b). In Fig. 1(a) and (c) the phase curves corresponding to the case $h = 0$ are represented by dashes. On these curves

$$|Q_2| \leqslant |\alpha R_1|^{1/3}, \quad P_2 \leqslant (27/32)^{1/6}|\alpha R_1|^{2/3} \tag{3.7}$$

3.3. *Action-angle variables in the approximate system.* In order to use the results of the KAM theory to prove Theorem 2, it is now convenient to introduce the variables I_i and w_i ($i = 1, 2$) into the system with Hamiltonian (3.2). These variables are action-angle variables [8] in the approximate system with Hamilton function (3.4). Hamilton function (3.2) can be written in the following form in the I_i, w_i variables

$$H = H^{(0)}(I_1) + \varepsilon^{1/2}H^{(1)}(I_1, I_2) + \varepsilon H^{(2)}(I_1, I_2, w_1, w_2; \varepsilon^{1/2}) \tag{3.8}$$

where $H^{(2)}$ is a 2π-periodic function in the angle variables w_1 and w_2.

If $\alpha = 0$ (in this case there are no terms of the third power in $|r_1|^{1/2}$, q_2, p_2 in (1.15)), we can obtain the replacement of variables $U_1, R_1, Q_2, P_2 \to w_1, I_1, w_2, I_2$, in explicit form

$$\begin{aligned} &U_1 = w_1, R_1 = I_1 \\ &Q_2 = -V^{1/3}\,\mathrm{cn}(v, k), \quad P_2 = 2V^{2/3}\,\mathrm{sn}(v, k)\,\mathrm{dn}(v, k) \\ &v = 2Kw_2/\pi, \quad V = 3\pi I_2/(4K) \end{aligned} \tag{3.9}$$

Here sn, cn and dn are the elliptic Jacobi functions, K is the complete elliptic integral of the first kind and the modulus k of the elliptic functions and of the integral is equal to $2^{-1/2}$.

Replacement (3.9) is canonical, univalent and 2π-periodic in w_2. We obtain that in Hamiltonian (3.8)

$$H^{(0)} = I_1, \quad H^{(1)} = (\tfrac{3}{4}\pi I_2 K^{-1})^{4/3} \tag{3.10}$$

while the function $H^{(2)}$ when $I_2 > 0$ is analytic with respect to all its arguments.

Suppose now that $\alpha \neq 0$. We will limit ourselves solely to obtaining action variables I_1 and I_2 Since, in the approximate system U_1 is a cyclical coordinate, we have $I_1 = R_1$. If $I_1 = 0$, the variable I_2 is introduced, as in the case when $\alpha = 0$, by the formulae from (3.9). Suppose $I_1 \neq 0$ and, to fix our ideas $\alpha I_1 > 0$ (the case when $\alpha I_1 < 0$ can be considered in a similar way). We then have

$$I_2 = \frac{1}{2\pi}\oint P_2 dQ_2 \tag{3.11}$$

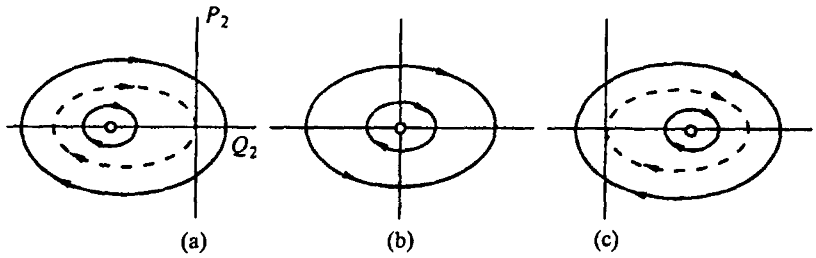

Fig. 1

where the integral is evaluated along the closed phase trajectory in Fig, 1*a*, specified by Eqs (3.3) and (3.6), in which $R_1 = I_1$.

If we make the replacement $Q_2 = (\alpha I_1)^{1/3}x$, in (3.11), we obtain

$$I_2 = \frac{\sqrt{2}\alpha I_1}{\pi} \int_{x_1}^{x_2} \sqrt{f(x)}dx \tag{3.12}$$

$$f = z - x - x^4, \quad z = (\alpha I_1)^{-4/3} h \ (z \geqslant -(27/256)^{1/3}) \tag{3.13}$$

The quantities $x_1, x_2 (x_1 \leqslant -2^{-2/3} \leqslant x_2)$ are the real roots of the equation $f = 0$. The other two roots of this equation are complex-conjugate umbers.

The derivative with respect to z of the right-hand side of (3.12) is non-zero. Hence, it can be solved for z. We obtain that z will be a certain function of the ratio I_2/I_1. We then obtain from (3.13) that $h = (\alpha I_1)^{4/3}\varphi\,(I_2/I_1)$. Consequently, in the Hamilton function (3.8) we have

$$H^{(0)} = I_1, \quad H^{(1)} = (\alpha I_1)^{4/3}\varphi(I_2 / I_1) \tag{3.14}$$

while the function $H^{(2)}$ is analytic with respect to its arguments when $I_1 \neq 0, I_2 > 0$.

3.4. *The change in the variables I_i (i = 1, 2) in the complete system and the orbital stability of the unperturbed periodic motion*. In the approximate system, described by Hamiltonian (3.4) the action variables are constant, $I_i(t) = I_i(0)$ (i = 1, 2). In the complete system, with Hamiltonian (3.2), the quantities I_i, generally speaking, will not be constant. But if the Hamiltonian of the complete system, written in the variables I_i, w_i in the form (3.8), satisfies the conditions

$$\frac{\partial H^{(0)}}{\partial I_1} \neq 0, \quad \frac{\partial H^{(1)}}{\partial I_2} \neq 0, \quad \frac{\partial^2 H^{(1)}}{\partial I_2^2} \neq 0 \tag{3.15}$$

then, by the KAM theory [7], for sufficiently small ε (i.e., by virtue of (3.1), in a sufficiently small neighbourhood of the trajectory of the unperturbed motion) for all initial conditions the variables $I_i(t)$ for all $t > 0$ will differ only slightly from their initial values

$$| I_i(t) - I_i(0) | < c\varepsilon^{1/2} (c - \text{const}), \quad i = 1, 2 \tag{3.16}$$

Consider the case when $\alpha I_1 = 0$. It immediately follows from expressions (3.10) for $H^{(0)}$ and $H^{(1)}$ that when $I_2 > 0$ conditions (3.15) are satisfied. Hence, for all $t > 0$ inequalities (3.16) hold. Hence, by virtue of the replacements (3.1) and (3.9) it follows that the periodic motion (1.1) is orbitally stable.

Suppose now that $\alpha I_1 \neq 0$. The first of conditions (3.15) is obviously satisfied, according to (3.14). For the derivatives of the function $H^{(1)}$ with respect to I_2, using [9] we obtain the following expression from (3.14) and (3.12)

$$\frac{\partial H^{(1)}}{\partial I_2} = \frac{(\alpha I_1)^{1/3}\pi\sqrt{2pq}}{2K(k)}, \quad \frac{\partial^2 H^{(1)}}{\partial I_2^2} = \frac{(\alpha I_1)^{-2/3}\pi^2\sqrt{pq}}{2K(k)} \frac{\partial}{\partial z}\left(\frac{\sqrt{pq}}{K(k)}\right)$$

$$p = [(x_1 + x_2)^2 + 2x_2^2]^{1/2}, \quad q = [(x_1 + x_2)^2 + 2x_1^2]^{1/2}$$

The modulus k of the complete elliptic integral of the first kind $K(k)$ is found from the equation

$$4pqk^2 = (x_1 - x_2)^2 - (p - q)^2$$

Note that the quantity $\partial H^{(1)}/\partial I_2$ is equal to the oscillation frequency corresponding to the closed trajectory in Fig. 1(a), divided by $\varepsilon^{1/2}$. This quantity is positive and, consequently, the second of conditions (3.15) is satisfied for all $z > -(27/256)^{1/3}$, i.e. for all closed trajectories in Fig. 1(a).

A check on the third of conditions (3.15) is more complicated. However, for the problem of the stability of periodic motion (1.1) it is sufficient to verify that this condition is satisfied at least for small positive values of z, to which the phase trajectories (3.6) surrounding the origin of coordinates $Q_2 = P_2 = 0$ correspond.

Calculations show that for small z the following estimate holds

$$\frac{\partial^2 H^{(1)}}{\partial I_2^2} = (\gamma + O(z))(\alpha I_1)^{-2/3}, \quad \gamma = \frac{\pi^2\sqrt{3}[(\sqrt{3}+1)K - 2\sqrt{3}E]}{3K^3} \tag{3.17}$$

where the modulus of the complete elliptic integrals K ad E is specified by the equality $4k = 2 - \sqrt{3}$. Since $\gamma \approx -1.372$, it follows from (3.17) that the third of inequalities (3.15) holds for sufficiently small z. Hence, for the set of all initial conditions which correspond to the trajectories surrounding the point $Q_2 = P_2 = 0$, lying close to the dashed trajectories (3.6) shown in Fig. 1(a), when $h = 0$, the variables $I_i(t)$ $(i = 1, 2)$ for all $t > 0$ satisfy inequalities (3.16), provided ε is sufficiently small. Hence, taking inequalities (3.7) and the replacement (3.1) into account, it follows that Theorem 2 also holds in the case when $\alpha I_1 \neq 0$.

4. THE STABILITY OF THE PERIODIC MOTION OF A DISC WHEN IT IS IN COLLISION WITH A HORIZONTAL PLANE

Suppose a thin uniform circular disc of radius R moves above a fixed absolutely smooth horizontal plane in a uniform gravitational field. From time to time the disc collides with the plane. The collisions are assumed to be absolutely elastic.

A motion of the disc exists when it rotates with constant angular velocity ω around its diameter, which occupies a vertical position, and then, as a result of the collisions, the disc periodically jumps a height h above the plane.

The orbital stability of this periodic motion of the disc (i.e. the stability with respect to perturbations of the angle of deviation of the plane of the disc from the vertical, the derivative of this angle with respect to time and the height the disc jumps above the plane) were investigated in [10]. It turned out that in the plane of the dimensionless parameters $a = \omega\sqrt{2h/g}$, $b = 4g/(\omega^2 R)$ there is a denumerable set of regions of stability and instability. In Fig. 2 we show part of the a, b plane containing the first two regions of instability, shown hatched in the figure. The values of the parameters a and b which lie on the curves separating the stability and instability regions were not considered in [10]. In Fig. 2 the first four such curves are denoted by $\gamma_k (k = 1, 2, 3, 4)$. The boundaries γ_2 and γ_4 are the vertical straight lines $a = \pi/2$ and $a = \pi$ respectively, while the boundaries γ_1 and γ_2 are specified by the equations $ab = \operatorname{tg} a$ and $ab = -\operatorname{ctg} a$.

Using the results of previous sections we investigated the stability of the periodic motion of a disc on the above-mentioned boundary curves. We were able to obtain the normalizing replacement (1.6) and the normal form (1.15) due to the fact that the fundamental matrix of the linearized equations of the perturbed motion can be written in explicit form [10]. Calculations showed that the quantity A, which occurs in Eq. (1.4), is equal to 1 on the boundaries γ_1 and γ_4 and -1 on the boundaries γ_2 and γ_3. The quantity δ in expression (1.15) is equal to 1 on the boundaries γ_1 and γ_3 and -1 on the boundaries γ_2 and γ_4. The coefficients h_{30} and h_{10} are identically equal to zero. An investigation of the sign of the coefficient h_{40} showed that it is positive along the whole curve γ_1; on γ_2 we have $h_{40} < 0$ when

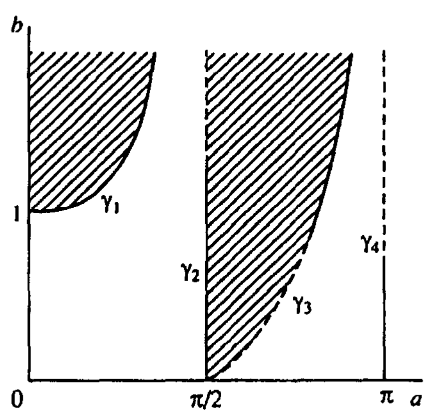

Fig. 2

$0 < b < 4/\pi$ and $h_{40} > 0$ when $b > 4/\pi$; on γ_3 when $\pi/2, < a < 2.543\ldots$ the coefficient h_{40} is negative, and when $2.543\ldots < a < \pi$ it is positive; on γ_4 we have $h_{40} < 0$ when $0 < b < 2/\pi$ and $h_{40} > 0$ when $b > 2/\pi$. On the basis of Theorems 1 and 2 we can therefore conclude that there is orbital stability and instability on the curves $\gamma_k (k = 1, 2, 3, 4)$. In Fig. 2 the stability and instability parts on these curves are represented by the continuous and dashed curves respectively.

5. THE CASE OF PARAMETRIC RESONANCE IN A SYSTEM CONTAINING A SMALL PARAMETER

Suppose Hamiltonian (1.2) depends on the parameter ε ($0 < \varepsilon \ll 1$), analytic with respect to it and, when $\varepsilon = 0$ is independent of the variable ξ_1. We will write the functions $\varphi_{nv}, \psi_{nv}, \chi$ from (1.3) in the form of series

$$\varphi_m = \sum_{k=0}^{\infty} \varepsilon^k \varphi_m^{(k)}(\xi_2,\eta_2,\xi_1),\quad \psi_m = \sum_{k=0}^{\infty} \varepsilon^k \psi_m^{(k)}(\xi_2,\eta_2,\xi_1),\quad \chi = \sum_{k=0}^{\infty} \varepsilon^k \chi^{(k)}(\xi_1) \tag{5.1}$$

$$\varphi_m^{(k)} = \sum_{\nu+\mu=m} \varphi_{\nu\mu}^{(k)}(\xi_1)\xi_2^\nu \eta_2^\mu,\quad \psi_m^{(k)} = \sum_{\nu+\mu=m} \psi_{\nu\mu}^{(k)}(\xi_1)\xi_2^\nu \eta_2^\mu$$

Here $\varphi_{\nu\mu}^{(k)}, \psi_{\nu\mu}^{(k)}, \chi^{(k)}$ are constant quantities if $k = 0$, and 2π-periodic functions if $k \geqslant 1$.

We will assume that when $\varepsilon = 0$ the periodic motion (1.1) is orbitally stable in the linear approximation, and the Hamiltonian $\varphi_2^{(0)}$ of system (1.5) has the form of the Hamiltonian of a harmonic oscillator of frequency ω

$$\varphi_2^{(0)} = \tfrac{1}{2}\omega(\xi_2^2 + \eta_2^2) \tag{5.2}$$

In this section, we will consider the case of parametric resonance, when the quantity 2ω is close to an integer n, and we will pay particular attention to the case of odd n.

5.1. *Normalization of Hamiltonian function (1.2) when $\varepsilon = 0$.* It is more convenient to carry out the investigation assuming that the Hamilton function (1.2) when $\varepsilon = 0$ is normalized to fourth-power terms inclusive with respect to $|\eta_1|^{1/2}, \xi_2, \eta_2$. Using the Depri-Hori method we can obtain the normalizing replacement of variables $\xi_i, \eta_i \to \xi_i^*, \eta_i^*$ as a canonical univalent transformation. In the variables ξ_i^*, η_i^* the Hamiltonian (1.2) when $\varepsilon = 0$ does not contain third-power terms in $|\eta_1^*|^{1/2}, \xi_2^*, \eta_2^*$, while the set of fourth-power terms depends on η_1^* and on the combination $\xi_2^{*2}, + \eta_2^{*2}$. Without dwelling on the details, we will merely note that in the normalization the variable η_1 remains unchanged ($\eta_1 = \eta_1^*$) while the variables ξ_1 and ξ_2, η_2 differ from ξ_1^* and ξ_2^*, η_2^*, by terms the power of which is no less than the second and third respectively.

In the variables ξ_i^*, η_i^* Hamiltonian (1.2) can be written in the form of the following series

$$\Gamma^* = \Gamma_2^* + \Gamma_3^* + \Gamma_4^* + \ldots + \Gamma_m^* + \ldots \tag{5.3}$$

where Γ_m^* is a form of power m in $|\eta_1^*|^{1/2}, \xi_2^*, \eta_2^*$, where

$$\Gamma_2 = \eta_1^* + \varphi_2^*,\quad \varphi_2^* = \frac{1}{2}\omega(\xi_2^{*2} + \eta_2^{*2}) + \sum_{k=1}^{\infty} \varepsilon^k \Gamma_2^{*(k)}(\xi_2^*, \eta_2^*, \xi_1^*) \tag{5.4}$$

$$\Gamma_3 = \sum_{k=1}^{\infty} \varepsilon^k \Gamma_3^{*(k)}(\eta_1^*, \xi_2^*, \eta_2^*, \xi_1^*) \tag{5.5}$$

$$\Gamma_4 = c_{20}\eta_1^{*2} + \frac{1}{2}c_{11}(\xi_2^{*2} + \eta_2^{*2})\eta_1^* + \frac{1}{4}c_{02}(\xi_2^{*2} + \eta_2^{*2})^2 + \sum_{k=1}^{\infty} \varepsilon^k \Gamma_4^{*(k)}(\eta_1^*, \xi_2^*, \eta_2^*, \xi_1^*) \tag{5.6}$$

The coefficients of the forms $\Gamma_m^{*(k)}$ in (5.4)–(5.6) are 2π-periodic in ξ_1^*. Then, for the second-order forms $\Gamma_2^{*(k)}$ the equalities $\Gamma_2^{*(k)} = \varphi_2^{(k)}(\xi_2^*, \eta_2^*, \xi_1^*)$ hold, where $\varphi_2^{(k)} (k = 1, 2, \ldots)$ are coefficients of ε^k in the first of series (5.1). The expressions for the forms $\Gamma_3^{*(k)}, \Gamma_4^{*(k)}$ are much longer and will not be written here, since they will not be needed later. The constant coefficients c_{ij} from (5.6) are expressed in terms of the coefficients of expansions (1.2) by the formulae

$$\begin{aligned}
c_{20} &= \chi^{(0)} - \tfrac{1}{2}\omega^{-1}(\psi_{10}^{(0)2} + \psi_{01}^{(0)2}) \\
c_{11} &= \psi_{20}^{(0)} + \psi_{02}^{(0)} - \omega^{-1}[\psi_{10}^{(0)}(3\varphi_{30}^{(0)} + \varphi_{12}^{(0)}) + \psi_{01}^{(0)}(3\varphi_{03}^{(0)} + \varphi_{21}^{(0)})] \\
c_{02} &= \tfrac{1}{2}(3\varphi_{40}^{(0)} + \varphi_{22}^{(0)} + 3\varphi_{04}^{(0)}) - \tfrac{3}{4}\omega^{-1}[5(\varphi_{03}^{(0)2} + \varphi_{30}^{(0)2}) + \\
&+\varphi_{21}^{(0)2} + \varphi_{12}^{(0)2} + 2(\varphi_{30}^{(0)}\varphi_{12}^{(0)} + \varphi_{03}^{(0)}\varphi_{21}^{(0)})]
\end{aligned} \tag{5.7}$$

5.2. *Further transformation of the Hamiltonian.* Suppose the quantity 2ω is close to an odd integer number $2n + 1$. We will put

$$2n + 1 - 2\omega = 2\varepsilon\alpha \tag{5.8}$$

where α is a quantity of the order of unity.

Remark. The results of this section can also be used in the case when 2ω is close to an even number. This will be the case, for example, if there are no third-order terms Γ_3 or they can be eliminated using a normalizing transformation.

We will consider an auxiliary linear Hamilton system of differential equations with Hamilton function φ_2^* from (5.4) and independent variable ξ_1^*. Using a linear real canonical univalent replacement of variables $\xi_2^*, \eta_2^* \to \tilde{\xi}_2, \tilde{\eta}_2$, analytic in ε, close to identical, 2π-periodic in ξ_1^*, this system can be converted to a system whose Hamilton function $\tilde{\varphi}_2(\tilde{\xi}_2, \tilde{\eta}_2, \xi_1^*)$ has the following form

$$\begin{aligned}
\tilde{\varphi}_2 &= \tfrac{1}{2}\lambda(\tilde{\xi}_2^2 + \tilde{\eta}_2^2) + \tfrac{1}{2}[\varkappa_1 \sin(2n+1)\xi_1^* - \varkappa_2 \cos(2n+1)\xi_1^*](\tilde{\xi}_2^2 - \tilde{\eta}_2^2) + \\
&+[\varkappa_1 \cos(2n+1)\xi_1^* + \varkappa_2 \sin(2n+1)\xi_1^*]\tilde{\xi}_2\tilde{\eta}_2
\end{aligned} \tag{5.9}$$

The quantities λ, $\varkappa_1$, $\varkappa_2$ can be represented in the form of converging series

$$\lambda = \tfrac{1}{2}(2n+1) - \varepsilon(\alpha - \lambda^{(1)}) + \varepsilon^2\lambda^{(2)} + \ldots, \quad \varkappa_i = \varepsilon\varkappa_i^{(1)} + \varepsilon^2\varkappa_i^{(2)} + \ldots, \quad i = 1,2 \tag{5.10}$$

where $\lambda^{(k)}$, $\varkappa_i^{(k)}$ are constants. In particular [5], we have

$$\lambda^{(1)} = \frac{1}{2\pi}\int_0^{2\pi}(\varphi_{20}^{(1)} + \varphi_{02}^{(1)})d\xi_1 \tag{5.11}$$

$$\begin{aligned}
\varkappa_1^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} [\varphi_{11}^{(1)} \cos(2n+1)\xi_1 - (\varphi_{02}^{(1)} - \varphi_{20}^{(1)})\sin(2n+1)\xi_1]d\xi_1 \\
\varkappa_2^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} [\varphi_{11}^{(1)} \sin(2n+1)\xi_1 + (\varphi_{02}^{(1)} - \varphi_{20}^{(1)})\cos(2n+1)\xi_1]d\xi_1
\end{aligned} \tag{5.12}$$

where $\varphi_{\nu\mu}^{(1)}$ are the coefficients of $\varphi_{\nu\mu}^{(1)}(\xi_1)$ of quadratic form $\varphi_2^{(1)}$ from (5.1).

If the replacement $\xi_2^*, \eta_2^* \to \tilde{\xi}_2, \tilde{\eta}_2$, which reduces the Hamiltonian $\varphi_2^*(\xi_2^*, \eta_2^*, \xi_1^*)$ to the form (5.9), is supplemented by the replacement $\xi_1^*, \eta_1^* \to \tilde{\xi}_1, \tilde{\eta}_1$, given by the formulae

$$\xi_1^* = \tilde{\xi}_1, \quad \eta_1^* = \tilde{\eta}_1 + \tilde{\varphi}_2(\tilde{\xi}_2, \tilde{\eta}_2, \tilde{\xi}_1) - \varphi_2^*(\xi_2^*, \eta_2^*, \tilde{\xi}_1)$$

where the arguments ξ_2^*, η_2^* of the function φ_2^* are expressed in terms of $\tilde{\xi}_2, \tilde{\eta}_2$ in accordance with the replacement $\xi_2^*, \eta_2^* \to \tilde{\xi}_2, \tilde{\eta}_2$, we obtain a close to identical canonical univalent transformation $\xi_i^*, \eta_i^* \to \tilde{\xi}_i, \tilde{\eta}_i$ of all four phase variables. This transformation reduces Hamilton function (5.3) to the following form

$$\tilde{\Gamma} = \tilde{\eta}_1 + \tilde{\varphi}_2(\tilde{\xi}_2, \tilde{\eta}_2, \tilde{\xi}_1) + \tilde{\Gamma}_3 + \tilde{\Gamma}_4 + \ldots + \tilde{\Gamma}_m + \ldots \tag{5.13}$$

where $\tilde{\varphi}_2$ is the function (5.9), and $\tilde{\Gamma}_m$ is a form of power m in $|\tilde{\eta}_1|^{1/2}$, $\tilde{\xi}_2$, $\tilde{\eta}_2$ with coefficients that are 2π-periodic in $\tilde{\xi}_1$, and $\tilde{\Gamma}_3$, $\tilde{\Gamma}_4$ have a structure specified by Eqs (5.5) and (5.6). Only in these equations instead of the forms $\Gamma_3^{*(k)}$, $\Gamma_4^{*(k)}$ there will be certain other forms $\tilde{\Gamma}_4^{(k)}$, $\tilde{\Gamma}_4^{(k)}$.

5.3. *The stability of periodic motion (1.1) at the boundary of the parametric resonance region.* We will introduce the following notation

$$\varkappa=(\varkappa_1^2+\varkappa_2^2)^{1/2},\ \beta=\varepsilon(\alpha-\lambda^{(1)})-\varepsilon^2\lambda^{(2)}-\ldots,\ a_{02}=c_{02}-c_{11}(n+1/2)+c_{20}(n+1/2)^2 \qquad (5.14)$$

Theorem 3. In the region $|\beta|<\varkappa$ periodic motion (1.1) is orbitally stable. At the boundary $|\beta|=\varkappa$ of this region, for sufficiently small ε, the periodic motion is orbitally stable if $\beta a_{02}<0$, and unstable if $\beta a_{02}>0$.

For the proof we first note that since ω is a semi-integer or close to semi-integer number, the quantities ω and 3ω are not close to integers. Hence the canonical, close to identical (identical when $\varepsilon=0$) 2π-periodic with respect to $\bar{\xi}_1$ replacement of variables $\bar{\xi}_i,\bar{\eta}_i\to\xi_i',\eta_i'$, obtained, for example, using the Depri-Hori method, in Hamiltonian function (5.13) may completely annul the third-order terms $\bar{\Gamma}_3$. The converted Hamiltonian Γ' will have the form

$$\Gamma'=\eta_1'+1/4(2n+1-2\beta)(\xi_2'^2+\eta_2'^2)+1/2\varkappa\cos2\theta(\eta_2'^2-\xi_2'^2)+\varkappa\sin2\theta\xi_2'\eta_2'+$$

$$+c_{20}\eta_1'^2+1/2c_{11}(\xi_2'^2+\eta_2'^2)\eta_1'+1/4c_{02}(\xi_2'^2+\eta_2'^2)^2+\sum_{k=1}^{\infty}\varepsilon^k\Gamma_4'^{(k)}(\eta_1',\xi_2',\eta_2',\xi_1')+\ldots \qquad (5.15)$$

$$2\theta=(2n+1)\xi_1'+\theta_*$$

The angle θ_* is defined by the equations $\varkappa_1=\varkappa\sin\theta_*$, $\varkappa_2=\varkappa\cos\theta_*$, and $\Gamma_4'^{(k)}$ are forms of the fourth power in $|\eta_1'|^{1/2}$, ξ_2', η_2' with coefficients that are 2π-periodic in ξ_1'.

We then make the canonical univalent replacement of variables $\xi_i',\eta_i'\to\xi_i'',\eta_i''$ specified by the equations

$$\xi_1'=\xi_1'',\ \eta_1'=\eta_1''-1/4(2n+1)(\xi_2''^2+\eta_2''^2) \qquad (5.16)$$
$$\xi_2'=\cos\theta\,\xi_2''+\sin\theta\,\eta_2'',\ \eta_2'=-\sin\theta\,\xi_2''+\cos\theta\,\eta_2''$$

In the ξ_i'',η_i'' variables the quadratic part of the new Hamiltonian does not contain the variable ξ_i'', while the Hamiltonian itself will be

$$\Gamma''=\eta_1''+1/2[(\varkappa-\beta)\eta_2''^2-(\varkappa+\beta)\xi_2''^2]+a_{20}\eta_1''^2+1/2a_{11}(\xi_2''^2+\eta_2''^2)\eta_1''+$$

$$+1/4a_{02}(\xi_2''^2+\eta_2''^2)^2+\sum_{k=1}^{\infty}\varepsilon^k\Gamma_4''^{(k)}(\eta_1'',\xi_2'',\eta_2'',\xi_1'')+O_5 \qquad (5.17)$$

$$a_{20}=c_{20},\ a_{11}=c_{11}-(2n+1)c_{20}$$

The quantity a_{02} is defined in (5.14), $\Gamma_4''^{(k)}$ are forms of the fourth power in $|\eta_1''|^{1/2}$, ξ_2'', η_2'' with coefficients that are 4π-periodic in ξ_i'', and O_5 are higher-power terms.

The Hamiltonian specified by the first three terms in expansion (5.17) corresponds to the linearized equations of the perturbed motion. When the inequality $|\beta|<\varkappa$ is satisfied the characteristic equation has a positive root. Hence, by Lyapunov's theorem on stability in the first approximation [6], the periodic motion (1.1) is orbitally unstable in the region $|\beta|<\varkappa$.

We will now consider the limits of the instability region when $\beta=\varkappa>0$ or $\beta=-\varkappa<0$. We will make the canonical univalent transformation $\xi_i'',\eta_i''\to u_i,v_i$, specified by the equations

$$\xi_1''=u_1,\ \eta_1''=v_1,\ \xi_2''=(2\varkappa)^{-1/2}v_2,\ \eta_2''=-(2\varkappa)^{1/2}u_2,\quad\text{if}\quad\beta=\varkappa \qquad (5.18)$$
$$\xi_1''=u_1,\ \eta_1''=v_1,\ \xi_2''=(2\varkappa)^{1/2}u_2,\ \eta_2''=(2\varkappa)^{-1/2}v_2,\quad\text{if}\quad\beta=-\varkappa$$

This replacement reduces Hamiltonian (5.17) to the form (1.13). Here $F_3=0$ while F_2 and F_4 are obtained from the terms of the second and fourth powers in (5.17) by replacing the quantities ξ_i'', η_i'' using formulae (5.18). Then, by using formulae (1.16) and (1.17) one can calculate the coefficients of the normalized Hamiltonian (1.15). We obtain

$$\delta=-\operatorname{sign}\beta,\ h_{30}=h_{10}=0,\ h_{40}=\varkappa^2a_{02}+O(\varepsilon^3) \qquad (5.19)$$

Since the quantity $\varkappa$ is of the order of ε, the sign of the coefficient h_{40} for sufficiently small ε is identical with the sign of a_{02}. For small ε the quantities δh_{40} and βa_{02} have opposite signs. Hence, by Theorems 1 and 2 we obtain that at the boundary of the parametric resonance region the unperturbed motion (1.1) is orbitally stable if $\beta a_{02} < 0$, and unstable if $\beta a_{02} > 0$.

Theorem 3 is proved.

6. THE STABILITY OF PENDULUM-TYPE ROTATIONS OF A RIGID BODY ABOUT A FIXED POINT

Consider the motion of a rigid body of weight mg about a fixed point O. Suppose $Oxyz$ is a system of coordinates rigidly connected to the body, the axes of which are directed along the principal axes of inertia of the body for the pint O, and A, B and C are the corresponding moments of inertia. We will assume that the centre of gravity G lies in the Oyz plane at a distance l from the point O, and the angle between the section OG and the Oz axis is equal to σ.

We will assume that the constant projection of the kinetic moment of the body onto the vertical is equal to zero. The equations of motion then allow of a partial solution, corresponding to the rotations of the body, for which the Oyz plane is in a fixed vertical plane, while the body rotates about the horizontal axis Ox like a physical pendulum. Suppose the mean angular velocity Ω of this rotation is sufficiently large so that the dimensionless quantity $\varepsilon = mgl/(A\Omega^2)$ can be taken as a small parameter.

The problem of the orbital stability of the above-mentioned plane rotations for small ε was investigated in [11]. In Fig. 3, in the plane of the parameters $b = B/A$ and $c = C/A$ in the physically permitted region $1 + b \geqslant c, b + c \geqslant 1, c + 1 \geqslant b$), we distinguish region 1, where the moment of inertia A with respect to the axis of rotation Ox has the mean value, and regions 2 and 3, where A is the greatest and least of the moments of inertia respectively. In region 1, for sufficiently small ε, plane rotation is unstable, while in regions 2 and 3, for small ε, there is orbital stability, apart from parametric resonance regions where the rotation of the body is unstable. In three-dimensional space of the parameters a, b, and ε the regions of instability are confined to surfaces which, when $\varepsilon = 0$, issue from the curve in the b, c plane specified by the equation

$$3bc - 4(b + c) + 4 = 0 \tag{6.1}$$

Parts of the branches of this curve, passing through regions 2 and 3, are shown in Fig. 3.

On the basis of the results of Section 5 of this paper and results from [11], we will consider the stability of the plane rotations of the body on the above-mentioned surfaces, which bound the parametric resonance regions for small ε.

The quantity ω from (5.2) is calculated from the formula

$$\omega = [b^{-1}c^{-1}(1-b)(1-c)]^{1/2}$$

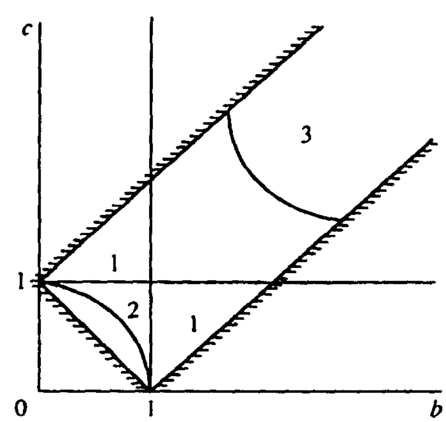

Fig. 3

On resonance curve (6.1) $\omega = 1/2$ and, consequently, we have $n = 0$ in Eq. (5.8). In the region of curve (6.1) the quantity α from (5.8) is positive in region 2 (Fig. 3) to the right of and above curve (6.1) and in region 3 to the left and below it. On the other hand, the quantity α is negative in region 2 to the left of and below this curve and in region 3 to the right of and above.

The coefficients c_{ij} (see (5.6) and (5.7)) are obtained as follows:

$$c_{20} = s/2, \quad c_{11} = s\omega, \quad c_{02} = -s[b^{-1}(1-b)+c^{-1}(1-c)]/4$$

where $s = 1$ in region 2 and $s = -1$ in region 3

The quantity $\lambda^{(1)}$ from (5.11) turned out to be zero, while for the quantities (5.12) we can obtain the following expressions

$$\varkappa_1^{(1)} = -(4b)^{-1}[2b - sr(2-b)]\sin\sigma, \quad \varkappa_2^{(1)} = s(4rc)^{-1}(3c-2)\cos\sigma \tag{6.2}$$

$$r = [c^{-1}(1-b)^{-1}b(1-c)]^{1/2}$$

In the Hamiltonian of the perturbed motion, reduced to the form (5.17), we have

$$\varkappa = \varepsilon\sqrt{\varkappa_1^{(1)2} + \varkappa_2^{(1)2}} + O(\varepsilon^2), \quad \beta = \varepsilon\alpha + O(\varepsilon^2), \quad a_{02} = -s\{1 + 2[b^{-1}(1-b)+c^{-1}(1-c)]\}/8 \tag{6.3}$$

It is easy to show that, near the resonance curve (6.1), the quantity a_{02} is negative both in region 2 and in region 3.

In the first approximation in ε, the parametric resonance region is given by the inequality

$$|\omega - 1/2| < \varepsilon\sqrt{\varkappa_1^{(1)2} + \varkappa_2^{(1)2}} \tag{6.4}$$

Bearing in mind the results of the analysis of the signs of the quantities α and a_{02} presented above, we obtain, on the basis of Theorem 3, that for small ε in the space b, c, ε on surfaces which bound the parametric resonance regions, the plane rotation of the body investigated is orbitally stable for values of b and c lying in region 2 to the right and above curve (6.1), and in region 3 to the left and below it, and unstable in region 2 to the left and below this curve and in region 3 to the right and above it.

The Kovalevskaya case. As an example we will consider the special case of a body, the geometry of the mass of which is close to the geometry of the mass of the body in the Kovalevskaya case, when $B = C = 2A$. In the Kovalevskaya case $b = c = 2$, and it is noteworthy that in this case exact resonance $\omega = 1/2$ occurs. This resonance denotes that the angular velocity of plane rotation of the body is exactly twice the frequency of small spatial oscillations of the body in the neighbourhood of this rotation.

The point $b = c = 2$ belongs to the part of curve (6.1) lying in region 3 (Fig. 3). Taking relations (6.2) and (6.3) into account we obtain that at this point

$$s = -1, \quad r = 1, \quad \varkappa = \varepsilon/2 + O(\varepsilon^2), \quad a_{02} = -1/8 \tag{6.5}$$

Suppose $b = 2 + \Delta b$, $c = 2 + \Delta c$, where Δb and Δc are small quantities. Then, up to first powers of Δb and Δc we have

$$\omega = 1/2 + 1/8(\Delta b + \Delta c)$$

Hence, we obtain from (6.3)–(6.5) that in the first approximation in ε the parametric resonance region is specified by the inequality

$$|\Delta b + \Delta c| < 4\varepsilon \tag{6.6}$$

On the boundary of region (6.6) the rotation of the body is orbitally stable for sufficiently small ε if the corresponding Δb and Δc are such that $\Delta b + \Delta c < 0$, and unstable if $\Delta b + \Delta c > 0$.

For example, if $\Delta b = 0$ and $\Delta c = 0$, then, in the first approximation in ε, the parametric resonance regions are specified respectively by the inequalities

$$2 - 4\varepsilon < c < 2 + 4\varepsilon \text{ and } 2 - 4\varepsilon < b < 2 + 4\varepsilon$$

At the boundaries $c = 2 - 4\varepsilon$ and $b = 2 - 4\varepsilon$ of these regions the rotation of the rigid body is orbitally stable, and on the boundaries $c = 2 + 4\varepsilon$ and $b = 2 + 4\varepsilon$ it is unstable.

This research was supported financially by the Russian Foundation for Basic Research (99-01-00405).

REFERENCES

1. BIRKOFF, G. D. *Dynamical Systems*. America Mathematics Society, New York, 1927.
2. IVANOV, A. P. and SOKOL'SKII, A. G., The stability of a non-autonomous Hamiltonian system for the fundamental type of parametric resonance. *Prikl. Mat. Mekh.*, 1980, **44**, 6, 963–970.
3. MARKEYEV, A.P., *Theoretical Mechanics*, CheRO, Moscow, 1999.
4. GIACAGLIA, G. E. O., *Perturbation Methods in Non-linear Systems*. Springer, New York, 1972.
5. MARKEYEV, A. P., *Libration Points in Celestial Mechanics and Space Dynamics*. Nauka, Moscow, 1978.
6. MALKIN, I. G., *The Theory of the Stability of Motion*. Nauka, Moscow, 1966.
7. ARNOLD, V. I., KOZLOV, V. V. and NEISHTADT, A. I., *Mathematical Aspects of Classical and Celestial Mechanics. Advances in Science and Technology Series: Modern Problems in Mathematics. Fundamental Trends*, Vol. 3. VINITI, Moscow, 1985.
8. GOLDSTEIN, H., *Classical Mechanics*. Addison-Wesley, Cambridge, MA, 1951.
9. GRADSHTEYIN, I. S. and RYZHIK, I. M., *Table of Integrals, Series and Products*. Academic Press, New York, 1980.
10. MARKEYEV, A. P., Investigation of the stability of the periodic motion of a rigid body when there are collisions with a horizontal plane. *Prikl. Mat. Mekh.*, 1994, **58**, 3, 71–81.
11. MARKEYEV, A. P., The plane and close-to-plane rotations of a heavy rigid body around a fixed point *Izv. Akad. Nauk SSSR. MTT*, 1988, 4, 29–36.

Translated by R.C.G.